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Riccati-Coupled Similarity Shock Wave Solutions for Multispeed Discrete Boltzmann Models

H. Cornille[1] **and T. Platkowski**[2]

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We study nonstandard shock wave similarity solutions for three multispeed discrete Boltzmann models: (1) the square $8v_i$ model with speeds 1 and $\sqrt{2}$ with the x axis along one median, (2) the Cabannes cubic $14v_i$ model with speeds 1 and $\sqrt{3}$ and the x axis perpendicular to one face, and (3) another $14v_i$ model with speeds 1 and $\sqrt{2}$. These models have five independent densities and two nonlinear Riccati-coupled equations. The standard similarity shock waves, solutions of scalar Riccati equations, are monotonic and the same behavior holds for the conservative macroscopic quantities. First, we determine exact similarity shock-wave solutions of coupled Riccati equations and we observe nonmonotonic behavior for one density and a smaller effect for one conservative macroscopic quantity when we allow a violation of the microreversibility. Second, we obtain new results on the Whitham weak shock wave propagation. Third, we solve numerically the corresponding dynamical system, with microreversibility satisfied or not, and we also observe the analogous nonmonotonic behavior.

KEY WORDS: Discrete Boltzmann models; Riccati equations; similarity shock wave solutions.

1. INTRODUCTION

For the densities N_i associated to the velocities v_i of the discrete Boltzmann models (DBMs[1,2]) the standard similarity shock waves (which we call Riccatian solutions)

$$N_i = n_{0i} + n_i/D, \qquad D = 1 + w, \qquad w = \delta e^{\gamma\eta}, \qquad \eta = x - \zeta t \qquad (1.1)$$

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[1] Service de Physique Théorique, CE Saclay, F-91191 Gif-sur-Yvette, France.
[2] Department of Mathematics, Informatics and Mechanics, Warsaw University, Warsaw, Poland.

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(n_{0i}, n_i, δ, γ, ζ being constants) are solutions of scalar Riccati equations, for instance, $N_{1,\eta} = aN_1^2 + bN_1 + c$. These densities are monotonic η-dependent functions. The same property holds for the macroscopic conservative quantities mass M, momentum J, and energy E, which are linear combinations of the densities. In that case, the nonmonotonic effects can exist only for nonconservative macroscopic quantities.

Is it possible, either for the microscopic densities or for the macroscopic conservative quantities, to observe nonmonotonic effects? Clearly we must enlarge the class (1.1).

In this study we answer this question for a family of multispeed DBMs,[3,4] Considering for the DBMs the restriction of their system of partial differential equations (PMEs) along the x axis and subtracting their linear conservation laws, we obtain PDEs with only one, two, three,..., independent nonlinear equations. The case of only one really nonlinear equation for a density N_1 leads to the scalar Riccati equation, which is linearizable and, taking into account the positivity of the densities, leads to the solutions (1.1). We discuss the important case of five independent densities and two coupled-Riccati nonlinear equations:

$$N_{i,\eta} = \sum_{k=1}^{2} (N_i d_{ik} + c_{ik}) N_k + f_i N_j^2 + e_i, \qquad i = 1, 2, \quad j \neq i \qquad (1.2a)$$

The standard linearization corresponds to the so-called[5] projective Riccati system ($f_j = 0$), which is impossible for the presently studied DBM because it requires that the front shock velocity ζ be equal to the velocity of one of the particles. Note that the type of solutions (1.1) is possible for the PDE (1.2a).[6] We propose[6] a new ansatz,

$$N_i = n_{0i} + (n_{1i} + wn_{2i})/D$$
$$D = 1 + \delta_1 w + \delta_2 w^2, \qquad w = e^{\gamma\eta}, \qquad \eta = x - \zeta t \qquad (1.2b)$$

which we call non-Riccatian). Let us consider two classes of DBMs with nonlinear equations of the type (1.2a): (1) mixing speeds models,[6] and (2) models recently called the class I hierarchy[4] with the same d-dependent system of PDEs. These DBMs include[3,4] the square $8v_i$ model with the x axis along one median, the $14v_i$ cubic Cabannes model with speeds 1 and $\sqrt{3}$, and another $14v_i$ model with speeds 1 and $\sqrt{2}$, which can be considered as a superposition of two $8v_i$ models. Solutions of the type (1.2b) and solutions of the standard Runge-Kutta procedure with nonmonotonic behavior have been obtained for the type (1) models[6] and the aim of this paper is to extend these results to the type (2) models.

In Section 2 and Appendix A1 we present the three physical models of class I type. The five densities M_1, N_1, R, M_2, and N_2 of this DBM are associated to the velocities whose projections along the $x = x_1$ axis have coordinates 1, 1, 0, -1, and -1. They satisfy the same 1D PDE with three linear conservation laws and two coupled nonlinear equations:

$$
\begin{gathered}
d=2,3, \qquad d_* = 2(d-1) \\
p_\pm = \partial_t \pm \partial_x, \qquad p_+ N_1 + p_- N_2 = 0, \qquad p_+ M_1 + p_- M_2 + d_* R_t = 0 \\
p_+ M_1 - p_- M_2 + 2d_* p_+ N_1 = 0, \qquad p_- N_2/\bar{\sigma}_1 = a M_2 N_1 - M_1 N_2 \\
R_t/\bar{\sigma}_2 = M_1 M_2 - R^2
\end{gathered} \tag{1.3}
$$

with $\bar{\sigma}_i$ proportional to the cross sections σ_i, $\bar{\sigma}_2 = 2\sigma_2/d$, and $a > 0$ a fixed number, but in general $a \neq 1$, which means that we allow a violation of the microreversibility.

In Section 3 we discuss different theoretical aspects of the class I hierarchy. First we study what we call the Rankine–Hugoniot (RH), relations which contain *both the three conservation laws for density functions of a similarity variable* η *with propagation speed* ζ

$$
N_i(\eta), \quad M_i(\eta), \quad R(\eta), \quad \eta = x - \zeta t, \quad i = 1, 2 \tag{1.4}
$$

and the four relations coming from the vanishing of the two collision terms for the two equilibrium states. These two states can be determined from both a scaling parameter called n_{01} and two arbitrary parameters, ζ and $a > 0$. Furthermore, three densities are linear combinations of two other, so that we can rewrite two of the nonlinear equations (1.3) as a coupled-Riccati differential system for the two remaining densities, chosen to be N_2 and R. Second, for the stability of the two equilibrium states, we generalize known weak shock results. In the Whitham[8] approach we allow $a \neq 1$, $a > 0$, and $\bar{\sigma}_i$ in (1.3) to be arbitrary. We find the sum of a fifth-, fourth-, and third-order differential operator. We show that the wave motions associated to the higher-order operators are exponentially damped at large time by the main waves (characteristic velocities) provided by the lower-order operator. Third, we study the Lax admissibility (established for conservation laws alone) criteria,[7,1] which allow the determination of the sound speeds associated to the upstream and downstream states. Requiring that both shock wave and sound wave are moving in the same direction in the upstream state, we show that this property holds in both upstream and downstream states. Finally, we show, by changing the n_{01} values, that a decrease as well as an increase of the local Boltzmann entropy (satisfying an H-theorem) across the shock are possible, without changing the parameters entering into the shock inequalities.

In Section 4 we determine non-Riccatian solutions (1.2b) which depend on one scaling parameter and two arbitrary parameters while ζ and $\bar{\sigma}_1/\bar{\sigma}_2$ are fixed. *We find two classes of solutions with either* N_i *of the Riccatian type or the five densities being non-Riccatian.* For the three models we give explicit examples with overshoots for one density M_1 and small undershoots for the energy E. We observe nonmonotonic effects only if $a-1>0$ is sufficiently large. For $a \simeq 1$ we get monotonic non-Riccatian solutions.

In Section 5 we consider the general similarity solution of (1.3) in the form (1.4), and reduce the original system to a 2D dynamical system. We find numerically the integral curve of this system, which gives the macroscopic shock profiles. The similarity solutions depend on ζ, on two parameters of the Maxwellian, and on $\bar{\sigma}_2$. We find the overshoot for M_1, not restricted to $a>1$, but also possible for $a \leqslant 1$. For the energy we observe a nonmonotonic behavior for $a>1$, which disappears when $a \to 1$.

2. CLASS I HIERARCHY OF MULTISPEED MODELS

In Appendix A1 we define the class I models by giving the coordinates (x_1, x_2) for $d=2$ and (x_1, x_2, x_2) for $d=3$ of the velocities with projections ± 1, 0 along the $x=x_1$ axis, which are associated to the five independent densities N_i, M_i, and R, $i=1, 2$.

The macroscopic conservative quantities mass M, momentum J, and energy E are linear combinations of the microscopic densities

$$\begin{aligned} M &= M_1 + M_2 + d_*(R + N_1 + N_2) \\ J &= M_1 - M_2 + d_*(N_1 - N_2) \\ E &= (M_1 + M_2 + d_*)/2 + d_* d_{**}(N_1 + N_2) \end{aligned} \tag{2.1}$$

$d_{**}=1$ and $d_{**}=3/2$ for the Cabannes model and we deduce the velocity $U=J/M$ and the pressure $P=2E-MU^2$.

With microreversibility satisfied we consider the Boltzmann-H functional for which we are able to prove an H-theorem and call Boltzmann entropy the associated $S=-H$. With microreversibility violated, we modify, as did Tartar,[9] the Boltzmann entropy, which satisfies an H-theorem. Introducing the relative entropy $S=-H$ with

$$H = \sum_{i=1}^{2} [d_* N_i \log(N_i/\alpha_i) + M_i \log(M_i/\beta_i)] + d_* R \log(R/\beta_0)$$

$$\alpha_2 \beta_1/\alpha_1 \beta_2 = a, \qquad \beta_0^2 = \beta_1 \beta_2$$

we find $[\partial_t + \partial_x(\cdots)]H \leqslant 0$. A simple choice is $\beta_i = \beta_0 = \alpha_2 = 1$, $\alpha_1 = 1/a$.

3. RH SOLUTIONS, WHITHAM AND LAX CONDITIONS, BOLTZMANN ENTROPY

3.1. Determination of the Two Equilibrium States

We assume that the densities (N_1, N_2, M_1, M_2, R) are functions of a similarity variable $\eta = x - \zeta t$, and define the two Maxwellian states corresponding to $|\eta| = \infty$:

$$\text{(i)} \quad (n_{01}, n_{02}, m_{01}, m_{02}, r_0), \qquad \text{(ii)} \quad (s_1, s_2, p_1, p_2, r_{00}) \tag{3.1}$$

with $s_i = n_{01} + n_{1i}$, $p_i = m_{0i} + m_{1i}$, $i = 1, 2$, $r_{00} = r_0 + r_1$. The three linear conservation laws (1.3) give relations for n_{1i}, m_{1i}, r_1:

$$y := (1-\zeta)/(1+\zeta), \qquad z := 2(1-\zeta)/\zeta d_*, \qquad n_{12} = y n_{11} \tag{3.2}$$

$$y m_{11} = m_{12} + 2 r_1 y/z, \qquad m_{11} + 2 d_* n_{11} + m_{12}/y = 0 \tag{3.3}$$

The two collision terms in (1.3) vanish for the two Maxwellian states:

$$a n_{01} m_{02} = n_{02} m_{01}, \qquad m_{01} m_{02} = r_0^2, \qquad a s_1 p_2 = s_2 p_1, \qquad p_1 p_2 = r_{00}^2 \tag{3.4}$$

$$\begin{aligned} a_{01} &:= a(n_{01} m_{12} + m_{02} n_{11}) - n_{02} m_{11} - m_{01} n_{12} \\ b_{01} &:= a n_{11} m_{12} - n_{12} m_{11} \end{aligned} \tag{3.5}$$

$$\begin{aligned} a_{02} &:= m_{01} m_{12} + m_{02} m_{11} - 2 r_0 r_1 \\ b_{02} &= m_{11} m_{12} - r_1^2 \\ a_{0i} + b_{0i} &= 0, \qquad i = 1, 2 \end{aligned} \tag{3.6}$$

For the determination of the two states (3.1) we have 11 parameters ζ, n_{ki}, m_{ki}, and r_k with $k = 0, 1$, $i = 1, 2$ ($a > 0$ is fixed), and seven independent relations. From the parameters $n_{01} > 0$ and n_{11} we define scaled variables $n_{k2} = n_{k1} \bar{n}_{k2}$, $m_{ki} = n_{k1} \bar{m}_{ki}$, $r_k = n_{k1} \bar{r}_k$, $k = 0, 1$, $a_{0i} = n_{01} n_{11} \bar{a}_{0i}$, $b_{0i} = n_{11}^2 \bar{b}_{0i}$, and rewrite the relations

$$\bar{m}_{02} = \bar{n}_{02} \bar{m}_{01}/a, \qquad \bar{r}_0 = (\bar{m}_{01} \bar{m}_{02})^{1/2} \tag{3.7}$$

$$\begin{aligned} \bar{m}_{12} &= -y(\bar{m}_{11} + 2 d_*), \qquad \bar{r}_1 = z(\bar{m}_{11} + d_*) \\ n_{11} &= -n_{01} \bar{a}_{0i}/\bar{b}_{0i}, \qquad i = 1, 2 \end{aligned} \tag{3.8}$$

Lemma 1. The two Maxwellian states can be determined from the knowledge of one scaling parameter n_{01}, two scaled parameters, and the propagation speed ζ:

$$n_{01} > 0, \quad \bar{m}_{01} > 0 \text{ arbitrary}, \quad \bar{n}_{02} > 0 \text{ arbitrary}, \quad \zeta \text{ arbitrary}, \ |\zeta| < 1 \tag{3.9}$$

First from (3.8) we obtain $\bar{m}_{02}$ and $\bar{r}_0$ and, with n_{01}, deduce the positive parameters of the Maxwellian (i). Second, in Appendix A2, from $\bar{a}_{01}\bar{b}_{02}=\bar{a}_{02}\bar{b}_{01}$, we show that $\bar{m}_{11}$ is the root of a cubic polynomial with coefficients determined by (3.9). Third, from the knowledge of $\bar{m}_{11}$ and (3.8) we deduce $\bar{m}_{12}$, $\bar{r}_1$ and $\bar{a}_{0i}$, $\bar{b}_{0i}$. Fourth, from $n_{11}=-n_{01}\bar{a}_{01}/\bar{b}_{01}$ we get n_{11} and consequently n_{1i}, m_{1i}, r_1 or finally the Maxwellian (ii).

3.2. Coupled-Riccati Equations for Two Densities (Appendix A3)

Corollary 1. The nonlinear equations (1.3) lead to a Riccati-coupled system for N_2 and R.

We first rewrite the two nonlinear equations (1.3) for similarity waves:

$$(1+\zeta)N_{2,\eta}/\bar{\sigma}_1=M_1N_2-aN_1M_2, \qquad \zeta R_\eta/\bar{\sigma}_2=R^2-M_1M_2 \quad (3.10)$$

N_1, M_1, and M_2 from the three linear conservation laws are linear combinations of N_2 and R, and substituted into (3.10) lead to a coupled Riccati system:

$$\begin{aligned} N_{2,\eta}&=d_{11}N_2^2+d_{12}N_2R+f_1R^2+c_{11}N_2+c_{12}R+e_1 \\ R_\eta&=f_2N_2^2+d_{21}N_2R+d_{22}R^2+c_{21}N_2+c_{22}R+e_2 \end{aligned} \quad (3.11)$$

The coefficients and the solutions of (3.11) *are functions of the arbitrary parameters* (3.9) *and of* σ_1/σ_2. For a projective Riccati system $f_2=0$ or $\zeta=-1$, which is impossible.

3.3. Whitham Weak Shock Wave Propagation (Appendix A4)

For the determination of the characteristic velocities it is usual to refer to the weak-shock Lax–Whitham theory.[7, 8] In the Whitham approach we study the stability of an equilibrium state when different linear differential order terms are present. How can wave motions defined by higher-order terms be exponentially damped by the main wave provided by the lower-order term? In the Lax approach we study the inequalities which must be satisfied by both the upstream and downstream states. We cannot apply directly the previous results[1] because here we assume that the σ_i are arbitrary and we allow a violation of the microreversibility. The Whitham approach was recently considered[8] for a $9v_i$ DBM where only the contribution of the main wave given by the lowest order operator was discussed and it is not clear that the higher waves do not modify the results.

For the present models, after linearization around a Maxwellian, we find the sum of a fifth-, fourth-, and third-order operator with associated polynomials P_5, P_4, and P_3. We must verify that the higher wave motions with speeds given by the $P_4=0$, $P_5=0$ roots are exponentially damped by the main wave motions (sound wave roots $\zeta^{(j)}$ of $P_3=0$). *While the P_5, P_3 roots are independent of σ_i, this is not true for the two $P_4=0$ roots.* In Lemma A1 we prove both that the five P_5 roots and the four P_4 roots are interlaced with two roots ± 1 having multiplicity 2 in P_5 and 1 in P_4, while the two other $\zeta^{\pm}$ roots of P_4 are of opposite sign and modulus less than 1 and that the three $\zeta^{(j)}$, $j=1, 2, 3$, roots of P_3 are real and belong to $]-1, 1[$. In Lemma A2 we prove that the P_4, P_3 roots are interlaced with the strict Whitham-like inequalities and consequently the wave motions with velocities $\zeta^{\pm}$ are exponentially damped for large time. Finally we prove (Lemma A3) that the wave motions with velocities ± 1, present in both P_5, P_4, are also exponentially damped for large time. The three weak-shock velocities $\zeta^{(j)}_{(i)}$, written in (A.7), for the (i) state can also be found (Appendix A5) from the Euler equations.

3.4. Lax Admissibility Criteria

From the above study there exist three different roots $\zeta^{(j)}_{(i)}$ associated to Maxwellian (i) and three other $\zeta^{(j)}_{(ii)}$ to Maxwellian (ii). However, we do not know whether, for instance, the (i) state is the upstream or downstream state. When the microreversibility is satisfied, Gatignol[1] has explained the application of the Lax[7] admissibility criteria to DBMs. Let us call $\zeta^{(j)}_{+\infty}$ the characteristic speeds associated to the states at $\pm\infty$. The Lax conditions are: for some index j, $1 \leqslant j \leqslant 3$, the two following inequalities hold:

$$\zeta^{(j)}_{\infty} < \zeta < \zeta^{(j)}_{-\infty}, \qquad \zeta^{(j-1)}_{-\infty} < \zeta < \zeta^{(j+1)}_{\infty} \tag{3.12}$$

Lemma 2. If the two Lax conditions are satisfied, then the j index is $j=2$.

Since j, $j\pm 1$ is present, only $j=2$ is possible. In DBMs, Gatignol[1] has presented a weaker condition with only the first condition satisfied and with also $j=1$ or 2. As we shall see, *the satisfaction of Lax conditions does not always guarantee the subsonic and supersonic inequalities.* Let us call ζ_{up} and ζ_{down} the characteristic velocities satisfying the first Lax inequality and corresponding to the upstream and downstream states, respectively, $V=\zeta-U$, and V_{up} and V_{down} the associated shock velocities and W_{up} and W_{down} the sound wave velocities:

$$W_{\text{up}} = V_{\text{up}} - \zeta + \zeta_{\text{up}}, \qquad W_{\text{down}} = V_{\text{down}} - \zeta + \zeta_{\text{down}}$$
$$V_{\text{up}} M_{\text{up}} = V_{\text{down}} M_{\text{down}} \tag{3.13}$$

We assume that the shock is physical if both the shock inequalities $|W_{\text{down}}| > |V_{\text{down}}|$ and $|W_{\text{up}}| < |V_{\text{up}}|$ are satisfied and only if at the upstream state (and at the downstream one) the shock wave and the sound wave move in the same direction. This means $V_{\text{up}} W_{\text{up}} > 0$ and $V_{\text{down}} W_{\text{down}} > 0$ and, from $V_{\text{up}} V_{\text{down}} > 0$, the four waves move in the same direction.

Lemma 3. If $\zeta_{\text{down}} = \zeta_\infty$, the shock inequalities are satisfied iff $V_{\text{up}} < 0$ and $W_{\text{up}} < 0$.

The Lax condition $\zeta_{\text{down}} < \zeta < \zeta_{\text{up}}$ implies $W_{\text{down}} < V_{\text{down}}$ and $W_{\text{up}} > V_{\text{up}}$. If $V_{\text{up}} > 0$, the supersonic inequality cannot be satisfied, so that $V_{\text{up}} < 0$ and $V_{\text{down}} < 0$ and the subsonic inequality $W_{\text{down}} < V_{\text{down}} < 0$ is satisfied. If $W_{\text{up}} < 0$, the supersonic inequality is also satisfied, while if $W_{\text{up}} > 0$, the two waves move in opposite direction at the upstream state.

Lemma 4. If $\zeta_{\text{up}} = \zeta_\infty$, the shock inequalities are satisfied iff $V_{\text{up}} > 0$ and $W_{\text{up}} > 0$.

The Lax condition $\zeta_{\text{up}} < \zeta < \zeta_{\text{down}}$ implies $W_{\text{up}} < V_{\text{up}}$ and $W_{\text{down}} > V_{\text{down}}$. If $V_{\text{up}} < 0$, the supersonic inequality is violated, while if $W_{\text{up}} < 0$, then $V_{\text{up}} W_{\text{up}} < 0$. We notice that the Maxwellian states depend on the scaling parameter n_{01}, while the parameters of the Lax conditions ζ, $\zeta^{(j)}_{\pm\infty}$ as well as V_{i}, V_{ii}, W_{i}, and W_{ii} are independent of n_{01}.

3.5. Local Boltzmann Entropy

Let us call S_{up} and S_{down} the two upstream and downstream S values, respectively, and S_{i} and S_{ii} the S values associated to the (i) and (ii) states, respectively. We find

$$-S_{\text{i}} = \sum_{j=1}^{2} [d_* n_{0j} \log(n_{0j}/\alpha_j) + m_{0j} \log(m_{0j})] + d_* r_0 \log(r_0), \qquad a\alpha_1 = \alpha_2 = 1$$

and

$$-S_{\text{ii}} = \sum_{j=1}^{2} [d_* s_j \log(s_j/\alpha_j) + p_j \log p_j] + d_* r_{00} \log(r_{00})$$

These asymptotic local entropy quantities are determined from the RH relations and n_{01} is a scaling parameter for the asymptotic densities n_{0i}, m_{0i}, r_0, s_i, p_i, and r_{00} (the scaled variables $\bar{n}_{02}$, $\bar{m}_{0i}$, $\bar{r}_0$, $\bar{s}_i$, $\bar{p}_i$, and $\bar{r}_{00}$ are independent of n_{01}), but not for S_{i} and S_{ii}. Let us define $\Delta S = S_{\text{down}} - S_{\text{up}}$, and $\Delta M = M_{\text{down}} - M_{\text{up}} = n_{01}\, \Delta\bar{M}$, with $\Delta\bar{M}$ a function of the scaled variables and independent of n_{01}. We find $\Delta S/n_{01} = -(\log n_{01})\, \Delta\bar{M} + \bar{F}$, with $\bar{F}$ constructed from the scaled parameters and independent of n_{01}. When n_{01} varies, the lhs (and also the rhs) has values from $-\infty$ up to $+\infty$ with only one zero for $n_{01}=n_{01,c}$. We assume a compressive (rarefactive) shock or $\Delta\bar{M}>0$ (<0). It follows that $\Delta S<0$ (>0) for $n_{01}\gg 1$ and $\Delta S>0$ (<0) for $n_{01}\ll 1$. *For rarefactive (compressive) shocks, there exists a critical value $n_{01,c}$ of the scaling parameter such that $\Delta S<0$ (>0) for $n_{01}<n_{01,c}$ and $\Delta S>0$ (<0) for $n_{01}>n_{01,c}$.*

4. NON-RICCATIAN EXACT SOLUTIONS (APPENDIX B)

4.1. New Relations for the Parameters (see Appendix B1)

The densities are of the non-Riccatian type (1.2b):

$$N_i=n_{0i}+(n_{1i}+wn_{2i})/D, \qquad M_i=m_{0i}+(m_{1i}+wm_{2i})/D$$
$$R=r_0+(r_1+wr_2)/D, \qquad D=1+\delta_1 w+\delta_2 w^2, \qquad w=e^{\gamma\eta}, \qquad \eta=x-\zeta t$$

When $|\eta|\to\infty$ the limits of the densities are the two states defined in (3.1). Consequently, the n_{ki}, m_{ki}, and r_k ($k=0, 1$; $i=1, 2$) satisfy the relations and the properties studied in Section 3. Here we introduce both eight new parameters γ, r_2, n_{22}, m_{2i}, and δ_i ($i=1, 2$), σ_2/σ_1 (with n_{21} as a scaling parameter) and nine new relations, or one more relation than parameters. *The solutions will depend on $\bar{m}_{01}$ and $\bar{n}_{02}$ as arbitrary parameters, and n_{01} and n_{21} as scaling parameters, while ζ will be found from a compatibility condition.*

The linear conservation laws give the relations (3.2)–(3.3) for the scaled parameters associated to n_{21}:

$$\bar{n}_{22}=n_{22}/n_{21}=y, \qquad m_{2i}=n_{21}\bar{m}_{2i}, \qquad r_2=n_{21}\bar{r}_2$$
$$\bar{m}_{22}=-y(\bar{m}_{21}+2d_*), \qquad \bar{r}_2=z(\bar{m}_{21}+d_*)$$

The two nonlinear equations (3.10) give six new relations:

$$\gamma(1+\zeta)/\bar{\sigma}_1=a_{11}/n_{22}=(b_{11}+a_{01}\delta_1)/(\delta_1 n_{12}-2n_{22})$$
$$=(b_{21}+\delta_2 a_{01})/(2n_{12}\delta_2-n_{21}\delta_1)$$
$$\gamma\zeta/\bar{\sigma}_2=a_{12}/r_2=(b_{12}+\delta_1 a_{02})/(\delta_1 r_1-2r_2)$$
$$=(b_{22}+\delta_2 a_{02})/(2r_1\delta_2-r_2\delta_1)$$

where the a_{1i} and b_{1i}, $i=1, 2$, defined in Appendix B1, are linear in the parameters n_{2i}, m_{2i}, and r_2, while b_{2i}, $i=1, 2$, are quadratic in these parameters.

4.2. Solutions As Functions of n_{01}, $\bar{m}_{01}$, $\bar{n}_{02}$ (see Appendix B)

From the parameters n_{k1}, $k=0, 1, 2$, we define scaled quantities $\delta_k = \bar{\delta}_k(n_{21}/n_{11})^k$, $a_{1i} = n_{01}n_{21}\bar{a}_{1i}$, and $b_{ki} = n_{k1}n_{21}\bar{b}_{ki}$, $k=1, 2$. We quote the main results. First (Lemma B1) we have two solutions: $\bar{\delta}_1 =$ and $\neq \bar{\delta}_2 + 1$ (respectively called Sol.B and Sol.A). Second (Lemma B2) and third (Lemma B3), $\bar{\delta}_1$ and $\bar{m}_{21}$ are known functions of the arbitrary parameters $\bar{m}_{01}$, $\bar{n}_{02}$, and ζ. Finally (Lemma B4) all parameters are constructed from the arbitrary ones, and ζ is fixed by a compatibility condition and n_{21} such that $\delta_1 = \bar{\delta}_1 \ (n_{21}/n_{11}) > 0$.

4.3. Properties of the Non-Riccatian Sol.B: $\bar{\delta}_1 = \bar{\delta}_2 + 1$

We define $\bar{w} := wn_{21}/n_{11}$, find $D = 1 + \bar{w}\bar{\delta}_1 + \bar{w}^2\bar{\delta}_2 = (1+\bar{w})(1+\bar{w}\bar{\delta}_2)$, and obtain

$$N_i = n_{0i} + n_{1i}/(1+\bar{w}\bar{\delta}_2)$$
$$M_i = m_{0i} + m_{1i}(1+\bar{w}\bar{m}_{2i}/\bar{m}_{1i})/D$$
$$R = r_0 + r_1(1+\bar{w}\bar{r}_2/\bar{r}_1)/D$$

N_i are monotonic solutions of the Riccatian type, while M_i and R are non-Riccatian only if $\bar{m}_{21} \neq \bar{m}_{11}$. (On the contrary, for $\bar{\delta}_1 \neq \bar{\delta}_2 + 1$, the five densities can be non-Riccatian.)

Lemma 5. For positivity we must have $\bar{w} > 0$ or $n_{21}/n_{11} > 0$, $\bar{\delta}_2 > 0$.

Lemma 6. Sufficient conditions for nonmonotonic behavior of the densities M_i and R are, respectively, $\delta_{M_i} = \bar{\delta}_1\bar{m}_{1i}/\bar{m}_{2i} < 1$ and $\delta_R = \bar{\delta}_1\bar{r}_1/\bar{r}_2 < 1$. The signs of the derivatives $M_{i,\eta}$ and R_η are given by

$$\bar{w}^2 + (\bar{\delta}_1\bar{m}_{1i}/\bar{m}_{2i} - 1) + 2\bar{w}\bar{m}_{1i}/\bar{m}_{2i}, \qquad \bar{w}^2 + (\bar{\delta}_1\bar{r}_1/\bar{r}_2 - 1) + 2\bar{w}\bar{r}_1/\bar{r}_2$$

4.4. Numerical Calculations

Figure 1 presents for (a) the $d=2$, (b) the $d=3$, $d_{**}=1$, and (c) the $d=3$, $d_{**}=3/2$ models numerical curves for positive non-Riccatian solutions (also see Table I). We observe (except Fig. 1c) similar features: overshoots for the microscopic density M_1, small undershoots for the energy E, and monotonic behavior for the mass M. All the densities (except M_1 in Figs. 1a and 1b) are monotonic. For $a \leqslant 1$ we have not found both positive densities and $\sigma_2 > 0$. For $a > 1$ we have obtained positivity, but only for $a - 1 > 0$ sufficiently large have we observed nonmonotonic effects. In order

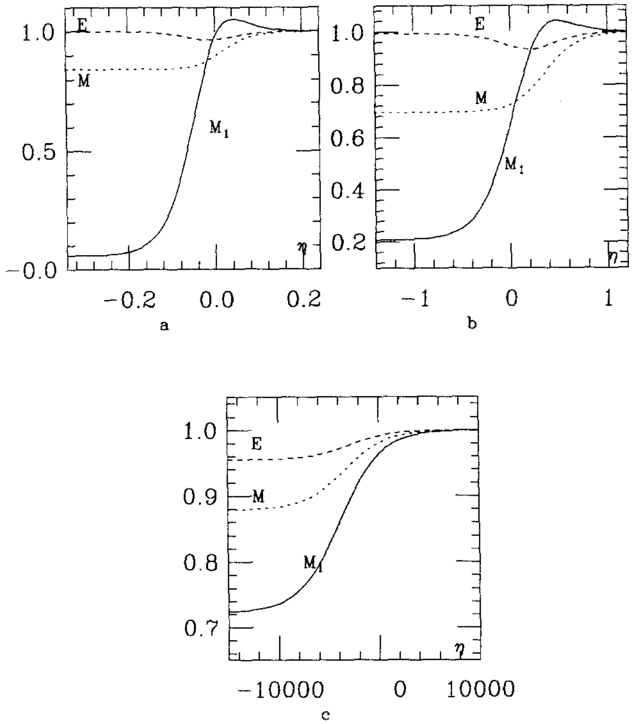

Fig. 1. Class I models, (a) $d=2$, (b) $d=3$, $d_{**}=1$, and (c) $d=3$, $d_{**}=3/2$.

to check the existence of M_1 overshoots, a useful criterion is provided by $\delta_{M_1}<1$ in Figs. 1a and 1b, and >1 in Fig. 1c.

For M, P, V, and W the corresponding values M_i, P_i, V_i, and $W_\mathrm{i}=V_\mathrm{i}-\zeta+\zeta_\mathrm{i}$ for state (i) and M_ii, P_ii, V_ii, and $W_\mathrm{i}=V_\mathrm{i}-\zeta+\zeta_\mathrm{ii}$ for state (ii) are positive. The shocks are rarefactive with M and P decreasing across the shock. In Figs. 1a–1c we find that the Lax index is $j=2$; for Figs. 1a and 1b the upstream state (ii) is at $+\infty$, which is confirmed by $\gamma<0$, while in Fig. 1c we find that the upstream state (i) is at $+\infty$, in agreement with

Table I

	a	$\bar{m}_{01}$	$\bar{n}_{02}$	$-\zeta$	$-\zeta_{(\mathrm{i})}$	$-\zeta_{(\mathrm{ii})}$	σ_1	σ_2	δ_{M_1}	γ	$n_{01,c}$
Fig. 1a	15.12	0.06	14.9	0.42	32×10^{-4}	0.75	1	2.6	0.79	−	2.1
Fig. 1b	8.72	0.45	7.7	0.25	-5×10^{-3}	0.57	$\sqrt{6/5}$	1.1	0.79	−	0.26
Fig. 1c	1.01	2.54	1.03	46×10^{-4}	5×10^{-3}	39×10^{-4}	1	3×10^{-6}	2.18	+	0.12

$\gamma > 0$. Due to $\zeta_{up} = \zeta_\infty$ (Lemma 4) and $V_i > 0$ and $W_i > 0$, the shock inequalities are satisfied.

We have calculated S_{up}, S_{down}, and $\Delta S = S_{up} - S_{down}$ when n_{01} is varying (in Figs. 1a and 1b $S_{ii} = S_{up}$; and $S_i = S_{up}$ in Fig. 1c). We find that the Boltzmann entropy decreases (or $\Delta S < 0$) for $n_{01} < n_{01,c}$ (see the $n_{01,c}$ values in Table I) and increases when $n_{01} > n_{01,c}$.

5. SOLUTION BY A DYNAMICAL SYSTEM APPROACH

In this section we solve numerically the 2D dynamical system (3.11), with the limit values corresponding to type (i) and type (ii) equilibria. The limit conditions are either $\lim(N_2, R) = (n_{02}, r_0)$ when $\eta \to \infty$ and (s_2, r_{00}) when $\eta \to -\infty$, or the inverse, and depend on which Maxwellian is at $\mp\infty$ (Section 3.4). In the general case, the free parameters of the problem are (see Lemma 1) $n_{01} = 1$, $\bar{m}_{01}$, $\bar{n}_{02}$, σ_2, and ζ.

We solve (1.3) using a Runge-Kutta fourth-order procedure, with the initial data $N_2 = n_{02} + \varepsilon_1$ and $R = r_0 + \varepsilon_2$, or $N_1 = s_2 + \varepsilon_1$ and $M_1 = r_{00} + \varepsilon_2$,

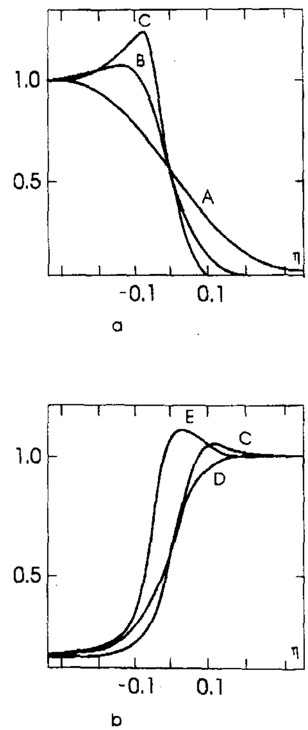

Fig. 2. M_1 profiles for $d = 2$.

ε_i small, depending on the direction of the integration. We start the integration from a neighborhood of the saddle, in a direction which can be calculated by the analysis of the eigenproblem for the linearized equations, and we end up in an arbitrary close neighborhood of the second singular point (node).

In Fig. 2a we present M_1 for different ζ values and the isotropic Maxwellian (i): $n_{01} = \cdots = r_0 = 1$, $d = 2$. Curves A, B, and C correspond, respectively, to $\zeta = 0.98$, 0.99, and 0.995. Note that M_1 overshoots if ζ is close to unity.

In Fig. 2b we present the influence of σ_2 on M_1. Curve C corresponds to $\sigma_2 = 1$, curve D to $\sigma_2 = 4$, and curve E to $\sigma_2 = 0.5$. The shock inequalities, and the common direction of the sound and shock waves in both equilibrium states, are satisfied, and the shocks are compressive. State (i) is at $+\infty$, and M and P increase from (i) to (ii).

We also note another family of solutions [compressive shocks, state (i) at $-\infty$], with $d = 3$, $\bar{m}_{01} = 0.1$, $\bar{n}_{02} = 10$, and $\zeta = -0.3$ with varying σ_2 and a. For $a = 10$ and $\sigma_2 = 10$ we observe an M_1 overshoot and an E undershoot. For $a = 10$ and $\sigma_2 = 1$ the M_1 overshoot disappears. For a decreasing to 1 the energy undershoot also disappears.

APPENDIX A. CLASS I MODELS, RH RELATIONS, WHITHAM INEQUALITIES, AND EULER WEAK-SHOCK RELATIONS

A1. Three Models of Class I

For the $d = 2$, $d = 3$, $\bar{\sigma}_1 = \sigma_1 \sqrt{6}/2$ and $d = 3$, $\bar{\sigma}_1 = \sigma_1 \sqrt{5}/2$ models we give the planar and spatial coordinates of the velocities associated to N_i, M_i, R:

$$N_1: (1, \pm 1), \quad N_2: (-1, \pm 1), \quad M_1: (1, 0), \quad M_2: (-1, 0)$$

$$R: (0, \pm 1), \quad \bar{\sigma}_1 = \sigma_1 \sqrt{5}/2, \quad N_1: (1, \pm 1, \pm 1), \quad N_2: (-1, \pm 1, \pm 1)$$

$$M_1: (1, 0, 0), \quad M_2: (-1, 0, 0), \quad R: (0, \pm 1, 0), (0, 0, \pm 1)$$

and

$$N_1: (1, \pm 1, 0), (1, 0, \pm 1), \quad N_2: (-1, \pm 1, 0), (-1, 0, \pm 1)$$

$$M_1: (1, 0, 0), \quad M_2: (-1, 0, 0), \quad R: (0, \pm 1, 0), (0, 0, \pm 1)$$

A2. Determination of $\bar{m}_{11}$ As a Function of the Arbitrary Parameters $\bar{m}_{01}$, $\bar{n}_{02}$, and ζ

From (3.2) and (3.8)

$$\bar{a}_{01}\bar{b}_{02} - \bar{a}_{02}\bar{b}_{01} = \sum_{k=0}^{3} (A_k - B_k \bar{m}_{11}^k) = 0$$

with $\bar{b}_{01} = a\bar{m}_{12} - y\bar{m}_{11}$, $\bar{b}_{02} = \bar{m}_{11}\bar{m}_{12} - \bar{r}_1^2$, $\bar{a}_{01} = a(\bar{m}_{12} + \bar{m}_{02}) - \bar{n}_{02}\bar{m}_{11} - y\bar{m}_{01}$, $\bar{a}_{02} = \bar{m}_{01}\bar{m}_{12} + \bar{m}_{02}\bar{m}_{11} - 2\bar{r}_0\bar{r}_1$. We define $A := ay + \bar{n}_{02}$, $Z = y + z^2$, and $\bar{m}_{11}$ is the root of a cubic polynomial. We have

$$\begin{aligned}
-\bar{a}_{01} &= \bar{m}_{11}A + y(2ad_* + \bar{m}_{01}) - a\bar{m}_{02} \\
-\bar{b}_{02} &= (\bar{m}_{11}^2 + 2d_*\bar{m}_{11})Z + z^2 d_*^2 \\
-\bar{a}_{02} &= \bar{m}_{11}(y\bar{m}_{01} + 2\bar{r}_0 z - \bar{m}_{02}) + 2d_*(y\bar{m}_{01} + \bar{r}_0 z) \\
-\bar{b}_{01} &= y(1+a)\bar{m}_{11} + 2ayd_* \\
A_3 &= AZ \\
A_2 &= Z[(2ad_* + \bar{m}_{01})y - a\bar{m}_{02}] + 2d_* ZA \\
A_1 &= Az^2 d_*^2 + 2d_* Z[y(2ad_* + \bar{m}_{01}) - a\bar{m}_{02}] \\
A_0 &= z^2 d_*^2 [y(2ad_* + \bar{m}_{01}) - a\bar{m}_{02}] \\
B_3 &= 0 \\
B_2 &= y(1+a)(y\bar{m}_{01} + 2\bar{r}_0 z - \bar{m}_{02}) \\
B_1 &= 2yd_*[y(1+2a)\bar{m}_{01} + z\bar{r}_0(1+3a) - a\bar{m}_{02}] \\
B_0 &= 4yad_*^2(y\bar{m}_{01} + \bar{r}_0 z)
\end{aligned} \tag{A.1}$$

A3. Riccati-Coupled Equations for N_2, R

We write N_1, M_1, M_2 as linear combinations of N_2, R:

$$\begin{aligned}
N_1 &= [(1+\zeta)N_2 + C_1]/(1-\zeta) \\
M_1 &= [\zeta(d-1)R - d_*(1+\zeta)N_2 + (C_2 + C_3)/2 - d_* C_1]/(1-\zeta) \\
M_2 &= -d_* N_2 - [\zeta(d-1)R + (C_2 - C_3)/2 + d_* C_1]/(1+\zeta)
\end{aligned} \tag{A.2}$$

Substituting into (3.10), we obtain the Riccati-coupled system (3.11) for N_2, R. The coefficients are functions of both the parameters (3.9) and of $\bar{\sigma}_i$, $i = 1, 2$. We have

$$
\begin{aligned}
d_{11} &= d_* \bar{\sigma}_1 (a-1)/(1-\zeta) \\
d_{12} &= \bar{\sigma}_1 (1+a)\, \zeta (d-1)/(1-\zeta^2) \\
f_1 &= 0 \\
d_{22} &= \bar{\sigma}_2 [1+\zeta^2 (d-1)^2/(1-\zeta^2)]/\zeta \\
d_{21} &= 0 \\
f_2 &= -\bar{\sigma}_2 d_*^2 (1+\zeta)/\zeta(1-\zeta)
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
c_{12} &= \bar{\sigma}_1 a\zeta (d-1) C_1/(1+\zeta)(1-\zeta^2) \\
c_{11} &= \bar{\sigma}_1 [C_2(1+a) + C_3(1-a) + 2d_* C_1(2a-1)]/2(1-\zeta^2) \\
e_1 &= \bar{\sigma}_1 a C_1 (C_2 - C_3 + 2d_* C_1)/2(1+\zeta)(1-\zeta^2) \\
c_{21} &= \bar{\sigma}_2 d_* (C_3 - 2d_* C_1)/\zeta(1-\zeta) \\
c_{22} &= \bar{\sigma}_2 (d-1)\, C_2/(1-\zeta^2) \\
e_2 &= \bar{\sigma}_2 [C_2^2 - (C_3 - 2d_* C_1)^2]/4\zeta(1-\zeta^2)
\end{aligned}
\tag{A.4}
$$

We note that $d_{11}=0$ if $a=1$, and $f_2=0$ if $\zeta=-1$.

A4. Generalization of the Weak-Shock Whitham Inequalities to Class I

In order to determine the sound wave velocities, we linearize the nonlinear equations around one equilibrium state and we obtain the sum of three differential operators with associated polynomials P_5, P_4, P_3 ($P_3=0$ for the characteristic velocities). In order to generalize the Whitham result (interlacing of the P_3, P_4 roots), we find well-defined inequalities between the P_5, P_4 roots and the P_4, P_3 roots. Then the main wave is provided by the P_3 roots (weak shocks), while the disturbances provided by the higher polynomials P_5, P_4 will be exponentially damped. We linearize Eqs. (1.3) around the (i) state: $N_i = n_{0i}[1+X_i(\eta_{(i)})]$, $M_i = m_{0i}[1+Y_i(\eta_{(i)})]$, $R = r_0[1+Y_0(\eta_{(i)})]$, $\eta_{(i)} = x - \zeta_{(i)} t$; and we define the operators $d_+ = n_{01} p_+$, $d_- = n_{02} p_-$, $\delta_+ = m_{01} p_+$, $\delta_- = m_{02} p_-$, $\delta_0 = r_0 \partial_t$, $z_1 = \bar{\sigma}_1 m_{01} n_{02}$, $z_2 = \bar{\sigma}_2 m_{01} m_{02}$. We obtain $\Delta Y(\eta_{(i)}) = 0$, Y being a column vector with components X_1, X_2, Y_1, Y_2, Y_0. Defining $\bar{\Delta} = \det(\Delta)$, we get

$$
2\bar{\Delta} = \begin{vmatrix}
d_+ & d_- & 0 & 0 & 0 \\
2d_* d_+ & 0 & \delta_+ & -\delta_- & 0 \\
0 & 0 & \delta_+ & \delta_- & d_* \delta_0 \\
-z_1 & d_- + z_1 & z_1 & -z_1 & 0 \\
0 & 0 & -z_2 & -z_2 & \delta_0 + 2z_2
\end{vmatrix}
\tag{A.5}
$$

This is the sum of fifth-order ($\bar{\Delta}_5$), fourth-order ($\bar{\Delta}_4$), and third-order ($\bar{\Delta}_3$) differential operators such that $\bar{\Delta}_k(h(\eta_{(i)}))=0$, $k=5, 4, 3$, for h an $\eta_{(i)}$-dependent function. We call $P_k(\zeta_{(i)})$ the associated $k=5, 4, 3$ polynomials. We find for the fifth-, fourth-, and third-order terms

$$\begin{aligned}
\bar{\Delta}_5 &= d_+ d_- \delta_+ \delta_- \delta_0 \\
B_5 &= n_{01} n_{02} m_{01} m_{02} r_0 \\
P_5 &= B_5 (1-\zeta_{(i)}^2)^2 \zeta_{(i)} \\
\bar{\Delta}_4 &= z_1 \delta_0 [\delta_- \delta_+ (d_+ + d_-) + d_* d_+ d_- (\delta_+ \delta_-)] \\
&\quad + z_2 d_+ d_- [2\delta_- \delta_+ + d_* \delta_0 (\delta_+ + \delta_-)/2] \\
P_4 &= (1-\zeta_{(i)}^2) P_2 \\
P_2 &= z_1 P_{21} + z_2 P_{22} = A_2 \zeta_{(i)}^2 + A_1 \zeta_{(i)} + A_0 \\
P_{21} &= \zeta_{(i)} r_0 [m_{01} m_{02} (n_{01}(1-\zeta_{(i)}) - n_{02}(1+\zeta_{(i)})) \\
&\quad + d_* n_{01} n_{02} (m_{01}(1-\zeta_{(i)}) - m_{02}(1+\zeta_{(i)}))] \\
P_{22} &= n_{01} n_{02} [2(1-\zeta_{(i)}^2) m_{01} m_{02} \\
&\quad + 0.5 d_* r_0 \zeta_i (m_{01}(1-\zeta_{(i)}) - m_{02}(1+\zeta_{(i)})]
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\bar{\Delta}_3 / z_1 z_2 &= 2 d_* d_+ d_- (\delta_+ + \delta_- + d_* \delta_0) \\
&\quad + (d_+ + d_-)[2\delta_+ \delta_- + d_* \delta_0 (\delta_+ + \delta_-)/2] \\
P_3 / z_1 z_2 &= 2 d_* n_{01} n_{02} (1-\zeta_{(i)}^2)[m_{01} - m_{02} - \zeta_{(i)}(m_{01} + m_{02} + d_* r_0)] \\
&\quad + [n_{01}(1-\zeta_{(i)}) - n_{02}(1+\zeta_{(i)})] \\
&\quad \times \{2 m_{01} m_{02} (1-\zeta_{(i)}^2) + d_* \zeta_{(i)} r_0 [m_{01}(1-\zeta_{(i)}) - m_{02}(1+\zeta_{(i)})]/2\}
\end{aligned} \tag{A.7}$$

with roots -1, 1 (multiplicity 2), and 0 for P_5, and -1, 1, and the two P_2 roots for P_4. We call $\zeta^{\pm}$ the $P_2=0$ roots; $\zeta^{(j)}$, $j=1, 2, 3$, the $P_3=0$ roots; $\zeta_0^{(j)}$, $j=1, 2$, the P_{21} roots ($P_2=0$, $z_2=0$); and $\zeta_\infty^{\pm}$ the P_{22} roots ($P_2=0$, $z_2=\infty$). The $\zeta^{(j)}$ roots are n_{01} independent and for the (i) state $n_{0i}=m_{0i}=r_1$ they are $(0, \pm(5/6)^{1/2})$, $(0, \pm(13/15)^{1/2})$ for $d=2, 3$.

A4.1. Whitham Conditions for the P_3, P_4 Roots

Lemma A1. (i) The two roots $\zeta^{\pm}$ of P_2 are real and belong to $]-1, 0[$ and $]0, 1[$. (ii) The three roots $\zeta^{(j)}$, $j=1, 2, 3$, of P_3 are real and belong to $]-1, 1[$. (iii) For $P_3(0)>0$, $\zeta^+ \in]\zeta_0^{(2)}, \zeta_\infty^+[$, while $\zeta^- \in]0, \zeta_\infty^-[$; and for $P_3(0)<0$, $\zeta^+ \in]0, \zeta_\infty^+[$, while $\zeta^- \in]\zeta_0^{(2)}, \zeta_\infty^-[$.

We recall that $a>0$, $n_{0i}>0$, $m_{0i}>0$, $r_0>0$, $r_0^2=m_{01}m_{02}$, $m_{01}n_{02}=am_{02}n_{01}$, $z_i>0$.

(i) From (A.6) we get $P_2(\pm 1)<0$, $A_0=P_2(0)>0$, $A_2<0 \rightarrow -1<\zeta^-<0<\zeta^+<1$.

(ii) We find $\zeta_0^{(1)}=0$, $\zeta_0^{(2)}=c_-/c_+$, $c_\pm := d_* n_{01}(an_{01}\pm n_{02})+m_{01}(n_{01}\pm n_{02})$, and

$$\zeta_0^{(2)}P_3(\zeta=0)>0, \qquad \zeta_0^{(2)}P_3(\zeta=\zeta_0^{(2)})<0$$
$$P_3(\pm 1)\gtrless 0, \qquad |\zeta_0^{(2)}|<1 \tag{A.8}$$

The P_3 roots $\in\,]-1,+1[$ if we find $-1<x_1<x_2<1$ with $P_3(x_1)>0$, $P_3(x_2)<0$. In the two cases $P_3(0)\gtrless 0$ we get $(x_1,x_2)=(0,\zeta_0^{(2)})$ and $(\zeta_0^{(2)},0)$.

(iii) We define $c:=(an_{01}-n_{02})/(an_{01}+n_{02})$, $|c|<1$, $\alpha := d_* r_0(m_{01}+m_{02})/4m_{01}m_{02}>0$, and write down some $P_{2i}(\zeta)$, $i=1,2$, properties: $P_{22}(0)>0$, $P_{2i}(\pm 1)<0$, and

$$\zeta(\zeta_0^{(2)}-\zeta)\,P_{21}(\zeta)>0, \qquad P_{22}(\zeta)[1-\zeta^2+\zeta(c-\zeta)\alpha]>0$$
$$-1<\zeta_\infty^-<0<\zeta_\infty^+<1 \tag{A.9}$$

We seek whether or not the $\zeta^\pm$ roots of P_2 belong to the positive or negative intervals limited by the corresponding P_{21}, P_{22} roots. We consider $P_3(0)\gtrless 0$ and, for simplicity, present the results for $P_3(0)>0$ with positive and negative ζ values for $P_{2i}(\zeta)$, $P_2(\zeta)$:

$$P_3(0)>0 \rightarrow \zeta_0^{(2)}>0$$
$$\zeta>\sup(\zeta_0^{(2)},\zeta_\infty^+) \rightarrow P_{21}<0,\ P_{22}<0,\ P_2<0$$
$$0<\zeta<\inf(\zeta_0^{(2)},\zeta_\infty^+) \rightarrow P_{21}>0,\ P_{22}>0,\ P_2>0$$
$$-1<\zeta<\zeta_\infty^- \rightarrow P_{21}<0,\ P_{22}<0,\ P_2<0$$

Lemma A2. The P_3, P_4 roots satisfy the Whitham inequalities

$$-1<\zeta^{(1)}<\zeta^-<\zeta^{(2)}<\zeta^+<\zeta^{(3)}<1 \tag{A.10}$$

In addition to (A.8), (A.9), and Lemma A1, we add

$$P_3(\zeta_\infty^\pm)\lesseqgtr 0 \qquad \text{for} \quad \zeta_\infty^\pm \gtrless 0$$

Due to the $P_3(\zeta)$ changes of signs for $\zeta=-1$, ζ^-, ζ^+, 1, it follows that (A.10) is satisfied.

A4.2. Application of the Whitham Method to Higher Waves

We rewrite (A.5):

$$2\bar{\Delta} = B_5 \partial_t (\partial_{t^2}^2 - \partial_{x^2}^2)^2 + B_4 (\partial_{t^2}^2 - \partial_{x^2}^2)(\partial_t + \zeta^+ \partial_x)(\partial_t + \zeta^- \partial_x)$$
$$+ B_3 \prod_{j=1}^{3} (\partial_t + \zeta^{(j)} \partial_x)$$

with $B_5 > 0$, $B_4 = -A_2 > 0$, and $B_3/z_1 z_2$, equal to

$$d_*(m_{01} + m_{02})[2n_{01} n_{02} + 0.5 r_0 (n_{01} + n_{02})]$$
$$+ 2d_*^2 r_0 n_{01} n_{02} + 2m_{01} m_{02}(n_{01} + n_{02}) > 0$$

We follow the Whitham principle that in a wave motion with speed ζ the derivatives ∂_t and $-\zeta\partial_x$ of any quantity are approximately equal. The main waves are provided by the third-order operator (weak shock motion). The disturbances produced by the fifth- and fourth-order operators must be, for large time, exponentially damped by the presence of the main waves. For the wave motions with velocities $\zeta^\pm$ of the fourth-order operator this is explained by Whitham.[8] Here we neglect the fifth-order operator, and a damping ($e^{\mu t}$, $\mu < 0$) occurs provided $B_3 B_4$ is positive and the inequalities (A.10) are satisfied. For instance, for a wave motion with velocity ζ^+ we find

$$\mu = B_3 \left[\prod (\zeta^{(j)} - \zeta^+) \right] \Big/ B_4 (\zeta^+ - \zeta^-)[(\zeta^+)^2 - 1] < 0$$

Let us now study the damping of the wave motions $f(x \mp t)$ associated to the fifth-order (multiplicity 2) and fourth-order (multiplicity 1) operators. As in Whitham,[8] with the same approximations, in such a wave motion $\partial_t \approx \mp \partial_x$ and a third-order $\partial_{x^3}^3$ differential operator (or $\mp \partial_{t^3}^3$) is factorized: $[d_2(\partial_t \pm \partial_x)^2 + d_1(\partial_t \pm \partial_x) + d_0]\partial_{t^3}^3$ with $d_2 = 4B_5 > 0$, $d_1 = -2P_2(\mp 1) > 0$, $d_0 = \mp z_1 z_2 P_3(\mp 1) > 0$. Seeking a damping factor when the time is large,

$$[d_2(\partial_t \pm \partial_x)^2 + d_1(\partial_t \pm \partial_x) + d_0]\, f(x \mp t)\, C(t) = 0$$

the solutions associated to $f(x - (-1)^j t)$ are $C = e^{\mu t}$, where $\mu < 0$ satisfy $d_2 \mu^2 + d_1 \mu + d_0 = 0$, or

$$2\mu^2 n_{0j} m_{0j} + \mu[2z_1(m_{0j} + d_* n_{0j}) + z_2 d_* n_{0j}] + d_* z_1 z_2 = 0, \qquad j = 1, 2 \quad \text{(A.11)}$$

with positive discriminants:

$$[n_{0j} z_2 d_* + 2z_1(d_* n_{0j} - m_{0j})]^2 + 16 z_1^2 m_{0j} n_{0j} d_* > 0, \qquad j = 1, 2$$

Lemma A3. The wave solutions $f(x \mp t)$ corresponding to the fifth- and fourth-order terms are exponentially damped at large times, with the decay coefficients $\mu < 0$ given by (A.11).

A5. Euler Weak-Shock Relations

In the Euler formalism we can also obtain the characteristic velocities as roots of a cubic polynomial. First we take into account in (3.7) and (3.8) the vanishing of two collision terms $a_{0i} + b_{0i} = 0$. Second we assume that n_{1i}, m_{1i}, r_1 are small, their products being negligible, or $a_{0i} \approx 0$. Third, from the linear conservation laws, n_{12}, m_{12}, r_1 are linear functions of n_{11}, m_{11} and the relations become

$$m_{11} a_{(i)} = n_{11} b_{(i)}$$

$$m_{11} c_{(i)} = n_{11} d_{(i)}$$

$$a_{(i)} = m_{02}\zeta(1+\zeta) - m_{01}\zeta(1-\zeta) - 4r_0(1-\zeta^2)/d_*$$

$$b_{(i)} = (1-\zeta)[2m_{01}\zeta d_* + 4r_0(1+\zeta)]$$

$$c_{(i)} = n_{01}[1 - \zeta + m_{02}(1+\zeta)/m_{01}]$$

$$d_{(i)} = m_{02}(1+\zeta) - (1-\zeta)\, n_{01}(2d_* + m_{02}/n_{02})$$

The compatibility condition $a_{(i)}d_{(i)} - c_{(i)}d_{(i)} = 0$ leads to the roots (A.7).

A6. Infinite-Mach-Number Shock

In the (ii) state, we assume that $s_1 = s_2 = p_1 = r_{00} = 0$ with only $p_2 \neq 0$. First we show that the (i) and (ii) states are positive if $1 > \zeta > 1/[1 + (d_*/2)^2/a]^{1/2} > 0$. From Section 3,

$$y = (1-\zeta)/(1+\zeta) = \bar{n}_{02} > 0$$

$$\bar{m}_{01} = \bar{m}_{11} = d_*/\{-1 + \zeta d_*/2[a(1-\zeta^2)]^{1/2}\} > 0$$

$$\bar{m}_{02} = \bar{n}_{02}\bar{m}_{01}/a = \bar{m}_{12} = -y(\bar{m}_{11} + 2d_*) > 0$$

$$r_0 = \bar{m}_{01}(\bar{n}_{02}/a)^{1/2} = \bar{r}_1 = z(\bar{m}_{11} + d_*) > 0$$

Second, for the determination of the $\zeta_{(ii)}$ at the (ii) state, we perform a limiting procedure in $P_3 = 0$ (with n_{0i}, m_{0i}, r_0 replaced by s_i, p_i, r_{00}). Let $s_i \to 0$, $p_1 \to 0$, $r_{00} \to 0$ with $p_2 = \text{cst}$. From $ap_2 s_1 = s_2 p_1$ it follows that $s_1/s_2 \to 0$, $s_1/\sqrt{p_1} \to 0$ and $2P_3/z_1 z_2 d_* s_2 p_2 r_{00} \simeq \zeta_{(ii)}(1+\zeta_{(ii)})^2$ and the two possible characteristic $\zeta_{(ii)}$ are 0, -1. The Lax condition gives $\zeta_{(ii)} = 0 < \zeta < \zeta_{(i)}$. Furthermore, $P_i > P_{ii} = 0$, $M_i - M_{ii} = 2\zeta(d_* n_{01} + m_{01})/(1+\zeta) + d_* r_{00} > 0$ and the *shocks are compressive with the* (ii) *upstream state at* $+\infty$.

APPENDIX B. NON-RICCATIAN SOLUTIONS DEPENDING ON n_{01} AND ARBITRARY $\bar{m}_{01}$, $\bar{n}_{02}$

For the non-Riccatian (1.2b) we have new parameters: $\bar{n}_{2i}$, $\bar{m}_{2i}$, $\bar{r}_2$, $\bar{\delta}_i$, $\bar{\sigma}_i$. Due to the existence of one linear relation with two densities, we find two classes of solutions which depend on the parameters (3.9), except ζ, which is fixed by a compatibility condition.

B1. Six New Relations Coming from the Nonlinear Equations

We substitute the non-Riccatian ansatz into (3.10), multiply by D^2 and define two w polynomials A_i at the lhs and B_i at the rhs:

$$B_i := a_{1i} + b_{1i} + \delta_1 a_{0i} + w(\delta_1 a_{1i} + \delta_2 a_{0i} + b_{2i}) + w^2 a_{1i}\delta_2$$

$$-\gamma(1+\zeta)(n_{22} - n_{12}\delta_1 - 2wn_{12}\delta_2 - n_{22}w^2\delta_2)/\bar{\sigma}_1 = A_1$$

$$-\gamma\zeta(r_2 - r_1\delta_1 - 2wr_1\delta_2 - r_2 w^2\delta_2)/\bar{\sigma}_2 = A_2$$

$$a(n_{k1}m_{22} + m_{k2}n_{21}) - n_{k2}m_{21} - m_{k1}n_{22} = a_{11} \text{ for } k=0, \text{ and } =: b_{11} \text{ for } k=1$$

$$b_{21} := an_{21}m_{22} - n_{22}m_{21}$$

$$m_{k1}m_{22} + m_{k2}m_{21} - 2r_k r_2 = a_{12} \text{ for } k=0, \text{ and } = b_{12} \text{ for } k=1$$

$$b_{22} := m_{21}m_{22} - r_2^2$$

Since the coefficients of w^0, w^1, w^2 are the same on the lhs and rhs, we obtain from w^2:

$$\bar{\sigma}_2/\bar{\sigma}_1 = \zeta a_{11} r_2/[(1+\zeta)a_{12}n_{22}], \qquad \gamma/\bar{\sigma}_1 = a_{11}/(1+\zeta)n_{22} \tag{B.1}$$

and four other relations. We define scaled variables with the scaling parameters n_{k1}, $k=0, 1, 2$:

$$\delta_i = (n_{21}/n_{11})^i\,\bar{\delta}_i, \qquad i=1,2$$

$$a_{1i} = n_{01}n_{21}\bar{a}_{1i}$$

$$b_{1i} = n_{11}n_{21}\bar{b}_{1i}$$

$$b_{2i} = n_{21}^2\bar{b}_{2i}$$

$$a(\bar{m}_{22} + \bar{m}_{k2}) - \bar{n}_{k2}\bar{m}_{21} - y\bar{m}_{k1} =: \bar{a}_{11} \text{ for } k=0, \text{ and } =: \bar{b}_{11} \text{ for } k=1$$

$$\bar{b}_{21} = a\bar{m}_{22} - y\bar{m}_{21} = \bar{b}_{11} - \bar{b}_{01}\bar{m}_{21}\bar{m}_{k2} + \bar{m}_{22}\bar{m}_{k1}$$

$$-2\bar{r}_k\bar{r}_2 =: \bar{a}_{12} \text{ for } k=0, \text{ and } = \bar{b}_{12} \text{ for } k=1$$

$$\bar{b}_{22} = \bar{m}_{21}\bar{m}_{22} - \bar{r}_2^2$$

Recalling that $n_{11} = -n_{01}\bar{a}_{0i}/\bar{b}_{0i}$ is not arbitrary, we write the four relations

$$\begin{aligned}\bar{\delta}_1 &= (2\bar{a}_{11}\bar{b}_{01} - \bar{a}_{01}\bar{b}_{11})/\bar{b}_{01}(\bar{a}_{11} - \bar{a}_{01}) \\ &= \bar{r}_2(2\bar{a}_{12}\bar{b}_{02} - \bar{a}_{02}\bar{b}_{12})/\bar{b}_{02}(\bar{r}_1\bar{a}_{12} - \bar{a}_{02}\bar{r}_2) \qquad \text{(B.2)}\end{aligned}$$

$$\begin{aligned}\bar{\delta}_2 &= (\bar{a}_{11}\bar{\delta}_1 - \bar{b}_{21}\bar{a}_{01}/\bar{b}_{01})/(2\bar{a}_{11} - \bar{a}_{01}) \\ &= \bar{r}_2(\bar{\delta}_1\bar{a}_{12} - \bar{b}_{22}\bar{a}_{02}/\bar{b}_{02})/(2\bar{r}_1\bar{a}_{12} - \bar{r}_2\bar{a}_{02}) \qquad \text{(B.3)}\end{aligned}$$

For the four equations (B.2)–(B.3) we have three unknown parameters $\bar{\delta}_1$, $\bar{\delta}_2$, $\bar{m}_{21}$ to be determined, and consequently ζ will be fixed by a compatibility condition.

Lemma B1. $(\bar{\delta}_1 - \bar{\delta}_2 - 1)(2\bar{a}_{11} - \bar{a}_{01}) = 0.$

From the first of relations (B.2)–(B.3) and $\bar{b}_{21} + \bar{b}_{01} = \bar{b}_{11}$, coming from the N_i linear relation, the result is trivial. We call Sol.A and Sol.B the solutions $2\bar{a}_{11} = \bar{a}_{01}$ and $\bar{\delta}_1 = \bar{\delta}_2 + 1$.

Lemma B2. $\bar{\delta}_1 = 2 + \bar{a}_{01}(1 + ay)/\bar{b}_{01}(ay + \bar{n}_{02})$ (also $\bar{\delta}_2$ for Sol.B) is a known function of the arbitrary parameters $\bar{m}_{01}$, $\bar{n}_{02}$, ζ of (3.9). We define $\bar{\delta}_m = \bar{m}_{21} - \bar{m}_{11}$.

In the first relation of (B.2) for $\bar{\delta}_1$, both numerator and denominator factorize $\bar{\delta}_m$:

$$\bar{a}_{01} - \bar{a}_{11} = \bar{\delta}_m(ay + \bar{n}_{02})$$

$$2\bar{a}_{11}\bar{b}_{01} - \bar{a}_{01}\bar{b}_{11} = \bar{\delta}_m[ay(\bar{a}_{01} - \bar{b}_{01}) + y(\bar{a}_{01} - a\bar{b}_{01}) - 2\bar{b}_{01}\bar{n}_{02}]$$

Lemma B3. $\bar{m}_{21}$ is a known function of the arbitrary parameters $\bar{m}_{01}$, $\bar{n}_{02}$, ζ, $\bar{m}_{21} = \bar{m}_{11} + \bar{a}_{01}/2(ay + \bar{n}_{02})$ for Sol.A and for Sol.B:

$$\begin{aligned}\bar{m}_{21} + d_* = \bar{\delta}_1\bar{b}_{02}d_*(\bar{m}_{02} + y\bar{m}_{01})/[&2\bar{b}_{02}(\bar{m}_{02} - y\bar{m}_{01} - 2\bar{r}_0 z) \\ &+ \bar{a}_{02}(-\bar{m}_{12} + y\bar{m}_{11} + 2z\bar{r}_1)]\end{aligned}$$

In Sol.A we use $\bar{a}_{11} - \bar{a}_{01} = -\bar{a}_{01}/2$. For Sol.B, in the last relation of (B.3) both numerator and denominator factorize $\bar{\delta}_m$:

$$\bar{r}_1\bar{a}_{12} - \bar{r}_2\bar{a}_{02} = \bar{\delta}_m z d_*(\bar{m}_{02} + y\bar{m}_{01})$$

$$\bar{a}_{12} - \bar{a}_{02} = \bar{\delta}_m(\bar{m}_{02} - y\bar{m}_{01} - 2\bar{r}_0 z)$$

$$\bar{b}_{12} - 2\bar{b}_{02} = \bar{\delta}_m(\bar{m}_{12} - y\bar{m}_{11} - 2z\bar{r}_1)$$

$$2\bar{a}_{12}\bar{b}_{02} - \bar{a}_{02}\bar{b}_{12} = 2\bar{b}_{02}(\bar{a}_{12} - \bar{a}_{02}) + \bar{a}_{02}(2\bar{b}_{02} - \bar{b}_{12})$$

$$\bar{r}_2 = \bar{b}_{02}\bar{\delta}_1(\bar{r}_1\bar{a}_{12} - \bar{a}_{02}\bar{r}_2)/(2\bar{a}_{12}\bar{b}_{02} - \bar{a}_{02}\bar{b}_{12}) = z(\bar{m}_{21} + d_*)$$

Lemma B4. From a compatibility condition we find ζ as a function of $\bar{m}_{01}$, $\bar{n}_{02}$.

From Lemmas B9 and B10, with $\bar{m}_{21}$ known as a function of ζ, $\bar{m}_{01}$, $\bar{n}_{02}$ we deduce $\bar{r}_2$, $\bar{m}_{22}$, $\bar{a}_{1i}$, $\bar{b}_{1i}$, $\bar{b}_{2i}$, $i=1$, 2, while $\bar{\delta}_1$ is known. For Sol.A the two last relations of (B.2)–(B.3) give both $\bar{\delta}_2$ and a compatibility condition for ζ. For Sol.B the first relation of (B.3) fixes ζ.

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